Cornell Potential Parameters for *S*-wave Heavy Quarkonia

Hee Sok Chung,[1] Jungil Lee,[1] and Daekyoung Kang[2]

[1]*Department of Physics, Korea University, Seoul 136-701, Korea*

[2]*Physics Department, Ohio State University, Columbus, Ohio 43210, USA*

Abstract

We compute derived quantities for various values of the model parameter of the Cornell potential model for the S-wave heavy quarkonia with radial quantum numbers $n = 1$, 2, and 3. Our results can be used to determine leading and relative-order-v^2 nonrelativistic quantum chromodynamics matrix elements for S-wave charmonia and bottomonia such as $\psi(2S)$, $\eta_c(2S)$, and $\Upsilon(nS)$ for $n = 1$, 2, and 3. These matrix elements will be essential ingredients for resumming relativistic corrections to processes involving those S-wave heavy quarkonium states.

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I. INTRODUCTION

One of the most interesting recent developments in heavy-quarkonium phenomenology is the introduction of a new technique for resumming relativistic corrections to S-wave quarkonium production and decay rates. The technique resums corrections to all orders in the heavy-quark velocity v in the heavy-quark-antiquark ($Q\bar{Q}$) rest frame [1, 2] within the color-singlet mechanism of the nonrelativistic quantum chromodynamics (NRQCD) factorization approach [3]. With the method, the relative-order-v^2 NRQCD matrix element for the S-wave charmonium, which has a power-ultraviolet divergence and needs subtraction, has been evaluated with improved accuracy [1, 2] compared with lattice calculations [4] and with the determination [5] obtained by using the Gremm-Kapustin relation [6]. In Ref. [1], the generalized version of the Gremm-Kapustin relation within the Cornell potential model [7] was derived. The generalized Gremm-Kapustin relation allows one to resum a class of relativistic corrections in a potential-model color-singlet $Q\bar{Q}$ wave function. The resummation method has also been applied to determine leading-order NRQCD matrix elements for the $1S$ charmonium states. The resultant values for the matrix elements are significantly greater than those known previously [1, 2]. In addition, the method has provided a reasonable solution [8] to the long-standing puzzle of the cross section for $e^+e^- \to J/\psi + \eta_c$ measured at the B-factories [9], which has been one of the greatest discrepancies between theory and experiment within the Standard Model.

Therefore, it is worthwhile to extend this method to study radially excited S-wave charmonia, $\psi(2S)$ and $\eta_c(2S)$, and spin-triplet S-wave bottomonia, $\Upsilon(nS)$, with radial quantum numbers $n = 1$, 2, and 3. Unfortunately, the potential-model parameters and derived quantities reported in Ref. [1] are only for the $1S$ and the $2S$ states. The tabulation of the parameters in Ref. [1] is not convenient to use in combination with the improved version of the resummation method reported in Ref. [2], which considered only the $1S$ charmonium states.

In this paper, we compute derived quantities for various values of the model parameter of the Cornell potential model for S-wave heavy quarkonia with radial quantum numbers $n = 1$, 2, and 3. The Cornell potential is a linear combination of the Coulomb and linear potentials. Following Ref. [7], we rescale the Schrödinger equation and solve the equation numerically for various values of the model parameter λ, which determines the strength

of the Coulomb potential relative to the linear potential. To achieve improved accuracies in calculating energy eigenvalues, we use the numerical method[1] given in Refs. [11, 12]. The potential-model parameters and derived quantities are listed as functions of λ. These parameters are ready for use to determine leading and relative-order-v^2 color-singlet NRQCD matrix elements for both S-wave charmonia and bottomonia.

The remainder of this paper is organized as follows. In Sec. II, we give a brief description of the Cornell potential model. Our numerical results are given in Sec. III, which is followed by a discussion in Sec. IV.

II. POTENTIAL MODEL

In order to compute leading and higher-order color-singlet NRQCD matrix elements for an S-wave heavy quarkonium with the radial quantum number $n = 1$, 2, or 3, we need to compute the binding energy ϵ_{nS} of the nS state that appears in the generalized Gremm-Kapustin relation [1, 6]. In this section, we describe briefly the Cornell potential model [7] that we use to compute ϵ_{nS}. We refer the reader to Refs. [1, 2] which contain more complete descriptions of the model in conjunction with the NRQCD factorization formalism and the resummation technique for relativistic corrections to quarkonium processes.

We employ the Cornell potential [7], which parametrizes the $Q\bar{Q}$ potential $V(r)$ as a linear combination of the Coulomb and linear potentials:

$$V(r) = -\frac{\kappa}{r} + \sigma r, \tag{1}$$

where κ is a model parameter for the Coulomb strength and σ is the string tension. The relation between the string tension σ and the corresponding parameter a in the original formulation of the Cornell potential model [7] is $a = 1/\sqrt{\sigma}$. By varying the parameters in the Cornell potential, one can obtain good fits to lattice measurements of the $Q\bar{Q}$ static potential [13]. Therefore, we assume that the use of the Cornell parametrization of the $Q\bar{Q}$ potential should result in errors that are much less than the order-v^2 errors (about 30% for a charmonium and about 10% for a bottomonium) that are inherent in the leading-potential approximation to NRQCD [1, 2].

[1] In previous calculations in Refs. [1, 10], the authors quoted the energy eigenvalues determined in Ref. [7]. A recently improved analysis given in Ref. [2] employed the numerical method used in the present work.

The Schrödinger equation for the S-wave radial wave function $R_{nS}(r)$ with the radial quantum number n is

$$\left[-\frac{1}{mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+V(r)\right]R_{nS}(r)=\epsilon_{nS}R_{nS}(r), \tag{2}$$

where m is the quark mass and ϵ_{nS} is the binding energy for the nS state. The model parameter m is distinguished from the heavy-quark mass m_Q that appears in the short-distance coefficients of NRQCD factorization formulas. For an S-wave state, the wave function is $\psi_{nS}(r)=R_{nS}(r)/\sqrt{4\pi}$.

The Schrödinger equation in Eq. (2) depends on the model parameters m and κ, where we assume that the string tension σ is common to both charmonium and bottomonium states and that the value for σ can be determined from lattice measurements of the $Q\bar{Q}$ static potential. Then, the dependence on the flavor appears through m and κ. Introducing the scaled radius ρ and scaled coupling λ [7],

$$\rho=(\sigma m)^{1/3}\,r, \tag{3a}$$

$$\lambda=\frac{\kappa}{(\sigma/m^2)^{1/3}}, \tag{3b}$$

which are dimensionless, one can rewrite the radial equation in Eq. (2) as [7]

$$\left[\frac{d^2}{d\rho^2}+\frac{\lambda}{\rho}-\rho+\zeta_{nS}\right]u_{nS}(\rho)=0, \tag{4}$$

where $u_{nS}(\rho)$ and ζ_{nS} are the dimensionless radial wave function and the dimensionless energy eigenvalue of the nS state, respectively. The relation between $R_{nS}(r)$ and $u_{nS}(\rho)$ is

$$R_{nS}(r)=\sqrt{\sigma m}\,\frac{u_{nS}(\rho)}{\rho}, \tag{5}$$

where the wave functions are normalized according to

$$\int_0^\infty |u_{nS}(\rho)|^2 d\rho=\int_0^\infty |R_{nS}(r)|^2 r^2 dr=1. \tag{6}$$

The binding energy is related to the dimensionless eigenvalue ζ_{nS} as

$$\epsilon_{nS}=[\sigma^2/m]^{1/3}\zeta_{nS}(\lambda). \tag{7}$$

Note that Eq. (4) depends only on λ. Therefore, the scaled equation can be solved for a given λ to get the wave function $u_{nS}(\rho)$ and the eigenvalue ζ_{nS}. The flavor dependence

appears when we invert them to get the radial wave function $R_{nS}(r)$ and the energy eigenvalue ϵ_{nS}. In this step, m looks independent of the model parameter λ. However, we can express m in terms of σ, λ, and the $1S$-$2S$ mass splitting [1, 2, 7]:

$$m(\lambda) = \sigma^2 \left[\frac{\zeta_{2S}(\lambda) - \zeta_{1S}(\lambda)}{m_{2S} - m_{1S}} \right]^3 . \tag{8}$$

For S-wave states, the wave function at the origin $\psi_{nS}(0) = R_{nS}(0)/\sqrt{4\pi}$ can be expressed as [1, 2, 7]

$$|\psi_{nS}(0)|^2 = \frac{m}{4\pi} \int d^3r |\psi_{nS}(r)|^2 \frac{\partial V(r)}{\partial r} = \frac{\sigma\, m(\lambda)}{4\pi} \left[1 + \lambda F_{nS}(\lambda)\right], \tag{9}$$

where $F_{nS}(\lambda)$ is the expectation value of $1/\rho^2$ for the nS state:

$$F_{nS}(\lambda) = \int_0^\infty \frac{d\rho}{\rho^2} \left| u_{nS}(\rho) \right|^2 . \tag{10}$$

For purposes of computation of the NRQCD matrix elements, it is convenient to express those matrix elements in terms of the potential-model parameters listed above. A convenient parametrization can be found in Ref. [2]. According to Ref. [2], the leading-order NRQCD matrix element for the S-wave heavy quarkonium depends on λ, $m(\lambda)$, and $F_{nS}(\lambda)$. The ratio of the relative-order-v^2 NRQCD matrix element to the leading-order one is proportional to $\zeta_{nS}(\lambda)$. The electromagnetic decay rate for the quarkonium, in which relativistic corrections to all orders in v are resummed, is expressed in terms of these NRQCD matrix elements [2]. Because $m(\lambda)$, $F_{nS}(\lambda)$, and $\zeta_{nS}(\lambda)$ depend on λ, the decay rate is completely determined by λ. Determination of the best value for λ can be made by imposing a requirement that the resummed formula for the decay rate should reproduce the measured rate. Therefore, the leading-order and the relative-order-v^2 NRQCD matrix elements are completely determined [2].

III. NUMERICAL RESULTS

In this section, we list our numerical values for the parameters and derived quantities for the S-wave heavy quarkonia with radial quantum numbers $n = 1$, 2, and 3. In order to provide a set of parameters that can be used for both charmonia and bottomonia, we list the values for ζ_{nS} and F_{nS} that are common to both cases. In Table I, we tabulate those values as functions of the model parameter λ, whose range has been chosen so that the parameters

TABLE I: Scaled energy eigenvalues ζ_{nS} and F_{nS} of the S-wave heavy quarkonium with radial quantum numbers $n = 1$, 2, and 3 as functions of the Coulomb strength parameter λ.

λ	ζ_{1S}	F_{1S}	ζ_{2S}	F_{2S}	ζ_{3S}	F_{3S}
0.0	2.338107	1.1248	4.087949	0.8237	5.520560	0.6983
0.1	2.253678	1.1869	4.029425	0.8525	5.473169	0.7178
0.2	2.167316	1.2532	3.970286	0.8821	5.425462	0.7375
0.3	2.078949	1.3237	3.910531	0.9125	5.377441	0.7576
0.4	1.988504	1.3989	3.850160	0.9437	5.329112	0.7780
0.5	1.895904	1.4789	3.789174	0.9756	5.280478	0.7987
0.6	1.801074	1.5641	3.727575	1.0083	5.231545	0.8196
0.7	1.703935	1.6546	3.665364	1.0417	5.182316	0.8408
0.8	1.604409	1.7507	3.602543	1.0758	5.132798	0.8622
0.9	1.502415	1.8527	3.539116	1.1105	5.082996	0.8837
1.0	1.397876	1.9608	3.475087	1.1459	5.032914	0.9054
1.1	1.290709	2.0753	3.410458	1.1818	4.982560	0.9272
1.2	1.180834	2.1965	3.345233	1.2183	4.931938	0.9492
1.3	1.068171	2.3246	3.279418	1.2552	4.881053	0.9712
1.4	0.952640	2.4599	3.213016	1.2927	4.829913	0.9933
1.5	0.834162	2.6026	3.146031	1.3306	4.778522	1.0155
1.6	0.712658	2.7529	3.078468	1.3689	4.726886	1.0377
1.7	0.588049	2.9110	3.010330	1.4077	4.675010	1.0598
1.8	0.460260	3.0773	2.941621	1.4468	4.622899	1.0820
1.9	0.329215	3.2518	2.872344	1.4862	4.570560	1.1041
2.0	0.194841	3.4348	2.802503	1.5260	4.517996	1.1262
2.1	0.057065	3.6264	2.732099	1.5662	4.465212	1.1482
2.2	−0.084182	3.8269	2.661134	1.6066	4.412212	1.1702
2.3	−0.228969	4.0364	2.589611	1.6474	4.359001	1.1921
2.4	−0.377362	4.2550	2.517529	1.6885	4.305582	1.2139
2.5	−0.529425	4.4829	2.444888	1.7299	4.251959	1.2356
2.6	−0.685221	4.7202	2.371688	1.7716	4.198135	1.2573
2.7	−0.844808	4.9670	2.297928	1.8137	4.144112	1.2788
2.8	−1.008244	5.2235	2.223605	1.8561	4.089893	1.3002
2.9	−1.175584	5.4897	2.148718	1.8989	4.035481	1.3216
3.0	−1.346882	5.7657	2.073261	1.9421	3.980877	1.3428

can be used to determine the NRQCD matrix elements for both charmonia and bottomonia. The dependence of the eigenvalues ζ_{nS} and F_{nS} on λ are shown in Figs. 1 and 2, respectively.

In Ref. [11], the authors solved the Schrödinger equation in Eq. (2) numerically by using the inverse iteration method introduced in Ref. [12]. The method uses a trial wave packet which is a linear combination of the eigenfunctions for a given system and the amplification operator of the form $(H - \zeta_{\rm trial})^{-1}$, where H is the Hamiltonian of the system and $\zeta_{\rm trial}$ is a trial eigenvalue. If this operator acts on an eigenfunction of the Hamiltonian, then the operator yields a multiplicative factor $(\zeta - \zeta_{\rm trial})^{-1}$, where ζ is the eigenvalue of that eigenfunction. As $\zeta_{\rm trial}$ gets closer to ζ, the multiplicative factor blows up. Therefore, if this operator acts on a wave packet, then an eigenfunction, whose eigenvalue is the nearest to $\zeta_{\rm trial}$, is selectively amplified. Therefore, by applying the operator repeatedly, one could obtain the eigenfunction and the corresponding eigenvalue from the expectation value of the Hamiltonian. By varying $\zeta_{\rm trial}$, one could also obtain the energy spectrum. The details of the numerical method can be found in Refs. [11, 12]. Here, we use this method to generate Table I.

If one substitutes $\zeta_{nS}(\lambda)$ listed in Table I and the measured $1S$-$2S$ mass splitting for either the charmonium or the bottomonium into Eq. (8), one can obtain the mass parameter $m(\lambda)$ as a function of λ. ϵ_{nS} and $\psi_{nS}(0)$ are calculable by substituting the parameters $\zeta_{nS}(\lambda)$, $m(\lambda)$, and $F_{nS}(\lambda)$ into Eqs. (7) and (9), respectively. A convenient way to determine the optimal value for the parameter λ is given in Refs. [1, 2].

IV. DISCUSSION

We have computed derived quantities for various values of the model parameter of the Cornell potential model for the S-wave heavy quarkonia with radial quantum numbers $n = 1$, 2, and 3. The scaled Schrödinger equation is solved numerically for various values for the model parameter λ which determines the strength of the Coulomb potential relative to the linear potential. The scaled energy eigenvalue ζ_{nS} and a derived potential quantity F_{nS} are computed as functions of λ. These numbers are useful in determining the leading and relative-order-v^2 NRQCD matrix elements for both S-wave charmonia and bottomonia with radial quantum numbers $n = 1$, 2, and 3. As an application, the leading-order NRQCD matrix elements for the $\Upsilon(nS)$ have already been calculated and used to determine the

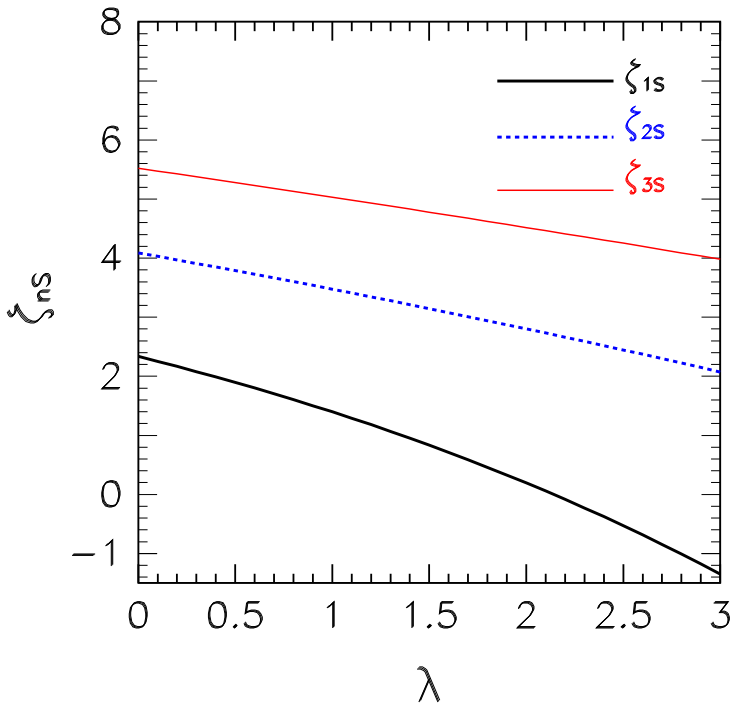

FIG. 1: Eigenvalues ζ_{nS} as functions of λ.

branching fractions and charm-momentum distributions for the inclusive charm production in $\Upsilon(nS)$ decays, which are being analyzed by the CLEO Collaboration [14]. The result listed in this paper can also be used to determine the NRQCD matrix elements for the $2S$ charmonium states. Once recent studies [15, 16, 17] to compute relativistic corrections to the leptonic width of the S-wave spin-triplet quarkonium are extended to complete the resummation of relativistic corrections to the process at next-to-leading order in α_s, our results can also be used to determine the leading-order and relative-order-v^2 NRQCD matrix elements for the S-wave quarkonium with accuracies better than the best available values in Ref. [2].

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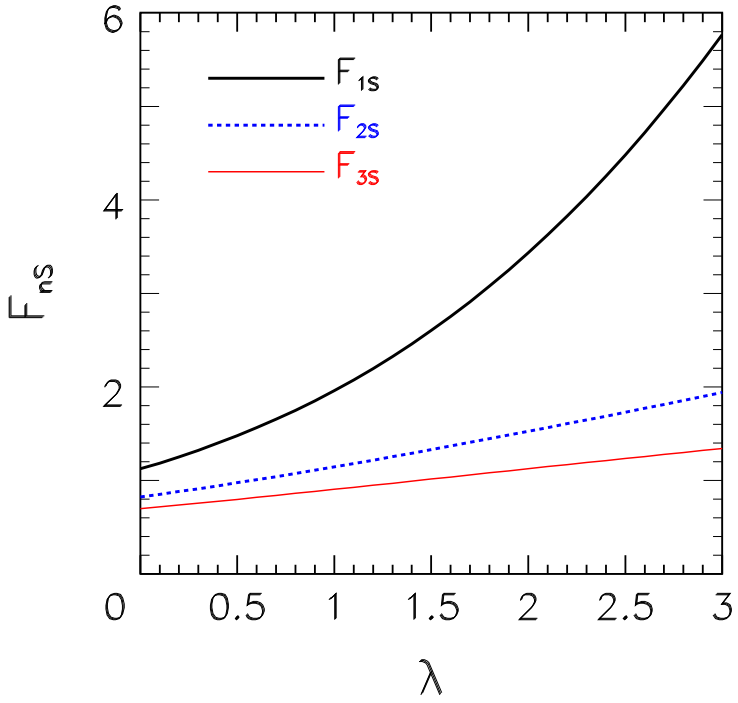

FIG. 2: F_{nS} as functions of λ.

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